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Self-similar solutions for a coupled system of nonlinear Schrödinger equations

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Abstract. This work is devoted to the study of self-similar solutions of the (2+1)-dimensional coupled nonlinear Schrödinger equations

$$\sqrt{-1}\,\psi_z^{(1)}+\psi_{xx}^{(1)}+\psi_{yy}^{(1)}+\eta[|\psi^{(1)}|^2+(1+h)|\psi^{(2)}|^2]\psi^{(1)}=0$$
$$\varepsilon\sqrt{-1}\,\psi_z^{(2)}+\psi_{xx}^{(2)}+\psi_{yy}^{(2)}+\eta[|\psi^{(2)}|^2+(1+h)|\psi^{(1)}|^2]\psi^{(2)}=0$$

where $\psi^{(1)}$ and $\psi^{(2)}$ are complex functions, h is a non-vanishing real parameter, $\eta=\pm 1$ and $\varepsilon=\pm 1$. We give the point-symmetry properties of the model and calculate generic (2+1)-dimensional symmetry reductions. Some exact and approximate solutions are obtained. In particular, we use a variational approach to determine and classify a set of physically relevant localized non-singular self-similar solutions.

1. Introduction

This work is devoted to a group theoretical analysis of the (2+1)-dimensional coupled nonlinear Schrödinger equations

$$\begin{aligned}&\sqrt{-1}\,\psi_z^{(1)}+\psi_{xx}^{(1)}+\psi_{yy}^{(1)}+\eta[|\psi^{(1)}|^2+(1+h)|\psi^{(2)}|^2]\psi^{(1)}=0\\&\varepsilon\sqrt{-1}\,\psi_z^{(2)}+\psi_{xx}^{(2)}+\psi_{yy}^{(2)}+\eta[|\psi^{(2)}|^2+(1+h)|\psi^{(1)}|^2]\psi^{(2)}=0\end{aligned}\qquad(1.1)$$

where $\psi^{(i)}(x,y,z)$ are complex functions (throughout the text, $i=1,2$), h is a non-vanishing real parameter, $\eta=\pm 1$ and $\varepsilon=\pm 1$.

This model is of particular interest in the field of transverse effects in nonlinear optics (a review and extensive bibliography can be found in [1] as part of a special issue on the subject). In fact, it is the basic model describing the time-independent copropagation ($\varepsilon=1$) [2] and counterpropagation ($\varepsilon=-1$) [3, 4] of two waves in self-focusing ($\eta=1$) or self-defocusing ($\eta=-1$) media. For the copropagating case, $\psi^{(1)}$ and $\psi^{(2)}$ denote the amplitudes of two circularly polarized waves and h can be normalized to unity. For the counterpropagating case, $\psi^{(1)}$ and $\psi^{(2)}$ stand for the forward and backward field amplitudes respectively with $0\leqslant h\leqslant 1$ describing the wavelength-scale index gratings in the medium that are due to the standing-wave interference pattern [4].

Our aim is to apply the techniques of Lie group theory in order to obtain similarity transformations that reduce (1.1) to algebraic or coupled ordinary differential equations (ODEs) and to solve some of them exactly, whenever possible, or approximately by using a variational approach.

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The so-called symmetry reduction method is a standard procedure [5-11] that has been applied to various partial differential equations. Let us mention, for instance, the cases of nonlinear relativistically invariant equations [12], the Kadomtsev–Petviashvili equation [13, 14], the classical ϕ^6-field equation [15-17], Davey–Stewartson equations [18], the 3D real Landau–Ginzburg equation [19, 20], nonlinear Schrödinger equations [21-27], the stimulated Raman scattering system [28] and the pumped Maxwell–Bloch system [29]. Briefly, the procedure consists of the following five steps.

(i) Find the Lie group G of point-symmetry transformations

$$\tilde{\boldsymbol{x}} = \Lambda_g(\boldsymbol{x}, \psi^{(1)}, \psi^{(2)}) \qquad \tilde{\psi}^{(i)}(\tilde{\boldsymbol{x}}) = \Omega_g^{(i)}(\boldsymbol{x}, \psi^{(1)}, \psi^{(2)}) \tag{1.2}$$

where $\boldsymbol{x} = (x, y, z)$, that leaves (1.1) invariant. In other words, if $\psi^{(1)}(\boldsymbol{x})$ and $\psi^{(2)}(\boldsymbol{x})$ are solutions of (1.1), so are $\tilde{\psi}^{(1)}(\tilde{\boldsymbol{x}})$ and $\tilde{\psi}^{(2)}(\boldsymbol{x})$.

(ii) Find subgroups of G for which their projected actions onto the space of independent variables (x, y, z) have orbits of codimension 0 and 1.

(iii) Find the invariants of the above subgroups and express the dependent variables in terms of them. Here we will make the following restrictions. First, we will consider only subgroups for which their actions on $(x, y, z, \psi^{(i)}, \psi^{(i)*})$ have orbits of codimension 4 or 5 in order to avoid any kind of partial reductions [7]. Second, we will restrict ourselves to generic $(2+1)$-dimensional reductions for which $\psi^{(i)}$ are explicit functions of the three independent variables; this excludes, for instance, solutions that can be obtained by 'rotating' the solutions of the $(1+1)$-dimensional version of (1.1). For the cases under consideration this provides expressions of the type

$$\psi^{(i)}(\boldsymbol{x}) = \alpha^{(i)}(\boldsymbol{x}) f^{(i)}(\xi(\boldsymbol{x})) \tag{1.3}$$

where α and ξ are known functions and the invariant ξ plays the role of the new independent variable.

(iv) Substitute the transformations (1.3) into the original equation in order to obtain the wanted algebraic equations or ODEs.

(v) Solve these reduced equations for $f^{(i)}(\xi)$ and substitute back into (1.3) to obtain solutions of (1.1) that are invariant under the considered subgroup of G.

The task of solving the reduced coupled ODEs is quite difficult in general. Usually, one restricts the analysis to the determination of conditions under which the reduced equations are of Painlevé type, that is, when none of their solutions have movable critical points. The method is well adapted for single equations since a large classification of second- and third-order Painlevé-type equations exists [30, 31]. This is not so easy for coupled systems.

Here we will give only few exact solutions. Rather, we will concentrate on a particular reduction and use a variational method to obtain the approximate expressions of a large class of localized non-singular solutions that are relevant in nonlinear optics. These self-similar solutions describe the transverse modal properties of a nonlinear Fabry–Perot interferometer [32]. They are invariant under a particular point-symmetry subgroup of the models that involves the Schrödinger conformal point-symmetry (also known as the Talanov lens transformation in optics [33]).

The variational approach we will use is a useful tool to obtain explicit approximate analytical solutions of nonlinear evolution equations. Moreover, it has recently been applied in a variety of problems in nonlinear optics [27, 32, 34-36]. Briefly, it consists of the following steps:

(i) reformulate the original evolution equation as a variational problem;

(ii) choose an appropriate trial function, with some free parameters in it, that describes the main characteristics of the solution;

(iii) solve the Euler–Lagrange equations for the chosen trial functions in order to determine the parameter values and to obtain the wanted approximate analytical solution.

The paper is organized as follows. In section 2, we show that the point-symmetry group G of (1.1) is the symmetry group of the (2+1)-dimensional Schrödinger equation, that is Sch(2), with an additional phase symmetry (an even larger group exists for the case $h=0$). The corresponding 10D symmetry algebra g is the direct sum between Sch(2) and a change of phase generator. The subalgebras of g are then quite easy to obtain from the known classification of non-conjugate subalgebras of Sch(2) [37]. In section 3, we give the list of the subalgebras we are interested in together with their corresponding invariants and the reduced equations they lead to. Some exact analytical solutions are given, whenever possible. Finally, in section 4, we use the variational approach to obtain the approximate solution behaviour of the physically relevant reduction for which exact analytical solution cannot be obtained. The calculations are supported with graphs showing some typical relations between the parameters involved in the approximate solutions.

2. Point-symmetry group

The point-symmetry group of the model (1.1) can be traced back to the point-symmetry group of the uncoupled case which is the 2D Schrödinger group. By adding the continuous symmetry transformation between $\psi^{(1)}$ and $\psi^{(2)}$, it is a straightforward calculation to show that (1.1) is invariant under the 10 following transformations:

three translations

$$\tilde{x}=x+x_0 \qquad \tilde{y}=y+y_0 \qquad \tilde{z}=z+z_0 \tag{2.1a, b, c}$$

two Galilean boosts

$$\tilde{\psi}^{(i)}=\psi^{(i)}\exp\left[\sqrt{-1}\,\varepsilon^{i-1}\frac{1}{2}\left(\frac{v_i^2}{2}z+v_ix\right)\right] \qquad \tilde{x}=x+v_iz \tag{2.2a}$$

$$\tilde{\psi}^{(i)}=\psi^{(i)}\exp\left[\sqrt{-1}\,\varepsilon^{i-1}\frac{1}{2}\left(\frac{v_i^2}{2}z+v_iy\right)\right] \qquad \tilde{y}=y+v_iz \tag{2.2b}$$

one rotation

$$\tilde{x}=x\cos\theta+y\sin\theta \qquad \tilde{y}=-x\sin\theta+y\cos\theta \tag{2.3}$$

one dilation

$$\tilde{z}=\frac{1}{a^2}z \qquad \tilde{x}=\frac{1}{a}x \qquad \tilde{y}=\frac{1}{a}y \qquad \tilde{\psi}^{(i)}=a\psi^{(i)} \tag{2.4}$$

one conformal transformation

$$\begin{gathered}\tilde{z}=\frac{z}{1-\lambda z} \qquad \tilde{x}=\frac{x}{1-\lambda z} \qquad \tilde{y}=\frac{y}{1-\lambda z}\\ \tilde{\psi}^{(i)}=\psi^{(i)}(1-\lambda z)\exp\left[\sqrt{-1}\,\varepsilon^{i-1}\frac{\lambda}{4}\frac{x^2+2y^2}{1-\lambda z}\right]\end{gathered} \tag{2.5}$$

and two constant changes of phase

$$\tilde{\psi}^{(i)}=\psi^{(i)}\exp(-\sqrt{-1}\,\varepsilon^{i-1}\phi_i) \tag{2.6}$$

where x_0, y_0, z_0, v_i, θ, a, λ and ϕ_i are the real parameters of the transformations. Note that a more general gauge invariance

$$\begin{pmatrix} \tilde{\psi}^{(1)} \\ \tilde{\psi}^{(2)} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix} \equiv M \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix} \tag{2.7}$$

where $M \in \mathrm{U}(2)$, exists when $h=0$ but we will not consider that limiting case here. When $\psi^{(2)}=0$, we recover the point-symmetry group of the scalar $(2+1)$-dimensional nonlinear Schrödinger equation. Finally, let us point out that relation (2.5) is also known in optics as the Talanov lens transformation [33]. It describes the image of a field produced by a thin spherical lens with focal length $1/\lambda$.

The corresponding generators of the above symmetry group G are respectively

$$P_x = \partial_x \qquad P_y = \partial_y \qquad P_z = \partial_z \tag{2.8}$$

$$K_x = zP_x - x\tfrac{1}{2}(M^{(1)} + M^{(2)}) \tag{2.9a}$$

$$K_y = zP_y - y\tfrac{1}{2}(M^{(1)} + M^{(2)}) \tag{2.9b}$$

$$J = yP_x - xP_y \tag{2.10}$$

$$D = 2zP_z + xP_x + yP_y - (\psi^{(1)}\partial_{\psi^{(1)}} + \psi^{(2)}\partial_{\psi^{(2)}} + \mathrm{cc}) \tag{2.11}$$

$$C = z^2P_z + zxP_x + zyP_y - \tfrac{1}{4}(x^2+y^2)(M^{(1)} + M^{(2)}) - z(\psi^{(1)}\partial_{\psi^{(1)}} + \psi^{(2)}\partial_{\psi^{(2)}} + \mathrm{cc}) \tag{2.12}$$

$$M^{(i)} = -\varepsilon^{i-1}\sqrt{-1}(\psi^{(i)}\partial_{\psi^{(i)}} - \mathrm{cc}). \tag{2.13}$$

They form a Lie algebra g with the commutation relations given in table 1. After the basis change $\{M^{(1)}, M^{(2)}\} \to \{M^{(1)} + M^{(2)}, M^{(1)} - M^{(2)}\}$, this algebra is identified as the direct sum of the Schrödinger algebra sch(2) and the generator $M^{(1)} - M^{(2)}$, that is

$$\mathrm{g} = \mathrm{sch}(2) \oplus \{M^{(1)} - M^{(2)}\}. \tag{2.14}$$

Since $\{M^{(1)} - M^{(2)}\}$ commutes with generators of sch(2), it is easy to determine the non-conjugate subalgebras of g by making use of the above direct sum decomposition as well as the known classification of non-conjugate subalgebras of sch(2) [37]. All splitting (with respect to the direct sum above) subalgebras of g are obtained by the union of subalgebras of sch(2) with $\{M^{(1)} - M^{(2)}\}$. All non-splitting ones are obtained by adding $\alpha(M^{(1)} - M^{(2)})$, where α is a real parameter, to each generator X_i of the subalgebras of sch(2) with the closure condition respected under mutual commutations. As stated in the introduction, the subalgebras of g we are interested in are those that

Table 1. Commutation relations for the generators of the algebra g.

	C	D	J	K_x	K_y	P_z	P_x	P_y	$M^{(1)}$	$M^{(2)}$
C	0	$-2C$	0	0	0	$-D$	$-K_x$	$-K_y$	0	0
D		0	0	K_x	K_y	$-2P_z$	$-P_x$	$-P_y$	0	0
J			0	K_y	$-K_x$	0	P_y	$-P_x$	0	0
K_x				0	0	$-P_x$	$(M^{(1)}+M^{(2)})/2$	0	0	0
K_y					0	$-P_y$	0	$(M^{(1)}+M^{(2)})/2$	0	0
P_z						0	0	0	0	0
P_x							0	0	0	0
P_y								0	0	0
$M^{(1)}$									0	0
$M^{(2)}$										0

have corresponding group orbits of codimension 4 or 5 in $(x, y, z, \psi^{(i)}, \psi^{(i)*})$ and that provide generic (2+1)-dimensional reductions. Under these restrictions, only eight non-conjugate subalgebras are left, that is

(1) $J+a_1M^{(1)}+a_2M^{(2)},\ D+b_1M^{(1)}+b_2M^{(2)},\ P_z$

(2) $K_x \mp P_y,\ C+P_z \pm J+b_1M^{(1)}+b_2M^{(2)}$

(3) $J+a_1M^{(1)}+a_2M^{(2)},\ D+b_1M^{(1)}+b_2M^{(2)}$

(4) $J+a_1M^{(1)}+a_2M^{(2)},\ C+P_z+b_1M^{(1)}+b_2M^{(2)}$

(5) $J+a_1M^{(1)}+a_2M^{(2)},\ P_z+b_1M^{(1)}+b_2M^{(2)}$

(6) $D+cJ+b_1M^{(1)}+b_2M^{(2)},\ P_z$

(7) $K_x,\ K_y+cP_y$

(8) $D+b_1M^{(1)}+b_2M^{(2)},\ K_y$

where a_i, b_i and c are real parameters. Only the first subalgebra generates a subgroup with orbit of codimension 0 in (x, y, z). All other reduce (1.1) to coupled ODEs as we will see in the next section. Even though subalgebras (7) and (8) do not appear explicitly in the classification given in [37], we have included them here. They do appear in the classification of subalgebras of the extended Galilei-similitude algebras, which is the Schrödinger algebra without the conformal symmetry [21].

3. Symmetry reductions

The determination of transformations that reduce the order of (1.1) proceeds through the calculation of the invariants of the subgroup G_0 of G [10]. We recall that if $\{X_j, j=1,2,\ldots,D\}$ is a basis for the D-dimensional Lie algebra of G_0, then these invariants are obtained by solving the equations

$$X_jQ(x, y, z, \psi^{(i)}, \psi^{(i)*})=0 \qquad (3.1)$$

where Q is an auxiliary function. For instance, subgroups with generic orbits of codimension 1 in (x, y, z) and 5 in $(x, y, z, \psi^{(i)}, \psi^{(i)*})$ lead to five invariants. For the case considered here, they can always be written in the form

$$I_1=\xi(x, y, z) \qquad I_i=f^{(i)}(\xi)=\psi^{(i)}(x, y, z)[\alpha^{(i)}(x, y, z)]^{-1} \qquad I_i^* \qquad (3.2)$$

where ξ and α are known functions and ξ plays the role of the new independent variable called the symmetry variable.

Substituting (3.2) into (1.1) reduces (1.1) to nonlinear coupled ODEs for $f^{(i)}$. Since $f^{(i)}$ are complex, one can make the substitution

$$f^{(i)}(\xi)=A^{(i)}(\xi)\exp[\sqrt{-1}\,\varepsilon^{i-1}\varphi^{(i)}(\xi)] \qquad (3.3)$$

in the reduced equations. It appears that one can always decouple the two real equations for the amplitude and phase of each wave and obtain $\varphi^{(i)}$ as a function of $A^{(i)}$. Let us run through the individual subgroups, identifying them by their Lie algebras, and give their corresponding reductions. Throughout this part, we have $r=(x^2+y^2)^{1/2}$ and $\theta=\tan^{-1}y/x$.

(1) $J+a_1M^{(1)}+a_2M^{(2)}, D+b_1M^{(1)}+b_2M^{(2)}, P_z$

This subalgebra provides a reduced algebraic equation that has the solutions, for $b_i=0$,

$$\psi^{(i)}=C_ir^{-1}\exp[\sqrt{-1}\,\varepsilon^{i-1}a_i\theta] \tag{3.4}$$

where

$$C_i^2=\frac{1}{\eta h}\frac{(1-a_i^2)-(1+h)(1-a_{3-i}^2)}{2+h} \qquad \text{for } h\neq 0,-2 \tag{3.5}$$

$$a_1^2=a_2^2 \qquad \eta(C_1^2+C_2^2)=a_1^2-1 \qquad \text{for } h=0 \tag{3.6}$$

$$a_1^2+a_2^2=2 \qquad \eta(C_1^2-C_2^2)=a_1^2-1 \qquad \text{for } h=-2. \tag{3.7}$$

(2) $K_x\mp P_y, C+P_z\pm J+b_1M^{(1)}+b_2M^{(2)}$

The symmetry variable for this subalgebra is

$$\xi=\frac{yz\pm x}{1+z^2} \tag{3.8}$$

and the reduction is

$$\psi^{(i)}=f^{(i)}(\xi)(1+z^2)^{-1/2}\exp\left[\sqrt{-1}\,\varepsilon^{i-1}\left\{\frac{1}{4z}[x^2+(z^2-1)\xi^2]-b_i\tan^{-1}z\right\}\right] \tag{3.9}$$

where

$$f_{\xi\xi}^{(i)}+(b_i-\xi^2)f^{(i)}+\eta[|f^{(i)}|^2+(1+h)|f^{(3-i)}|^2]f^{(i)}=0. \tag{3.10}$$

Equation (3.10) has the form of two coupled nonlinear quantum harmonic oscillators. For $f^{(i)}$ real, an approximate solution for the nonlinear modes can be obtained using the variational approach with the Hermite-Gauss polynomials as trial functions [27]. We will not go further into this analysis here.

(3) $J+a_1M^{(1)}+a_2M^{(2)}, D+b_1M^{(1)}+b_2M^{(2)}$

This subalgebra leads to the symmetry variable

$$\xi=rz^{-1/2} \tag{3.11}$$

and to the reduction

$$\psi^{(i)}=f^{(i)}(\xi)z^{-1/2}\exp\left[\sqrt{-1}\,\varepsilon^{i-1}\left(a_i\theta-\frac{b_i}{2}\ln z\right)\right] \tag{3.12}$$

where

$$f_{\xi\xi}^{(i)}+\left(\frac{1}{\xi}-\frac{\sqrt{-1}}{2}\xi\right)f_\xi^{(i)}+\left[\tfrac{1}{2}(b_i-\sqrt{-1})-\frac{a_i^2}{\xi^2}\right]f^{(i)}$$
$$+\eta[|f^{(i)}|^2+(1+h)|f^{(3-i)}|^2]f^{(i)}=0. \tag{3.13}$$

As for the uncoupled case, this is relevant in the description of the field envelopes in the early-stage evolution of a self-focusing collapse [38-40].

(4) $J+a_1M^{(1)}+a_2M^{(2)}, C+P_z+b_1M^{(1)}+b_2M^{(2)}$

The invariants of this subalgebra yield

$$\xi = r(1+z^2)^{-1/2} \tag{3.14}$$

and the reduction

$$\psi^{(i)} = f^{(i)}(\xi)(1+z^2)^{-1/2} \exp[\sqrt{-1}\,\varepsilon^{i-1}(\tfrac{1}{4}z\xi^2 + a_i\theta - b_i \tan^{-1} z)] \tag{3.15}$$

where

$$f^{(i)}_{\xi\xi} + \frac{1}{\xi} f^{(i)}_{\xi} + \left(b_i - \tfrac{1}{4}\xi^2 - \frac{a_i^2}{\xi^2}\right) f^{(i)} + \eta[|f^{(i)}|^2 + (1+h)|f^{(3-i)}|^2] f^{(i)} = 0. \tag{3.16}$$

Equation (3.16) describes a nonlinear coupling between the 'radial' parts of isotropic 2D quantum harmonic oscillators. On the other hand, this reduction is also of particular interest in nonlinear optics where it describes the transverse modal field in a nonlinear Fabry-Perot interferometer with spherical mirrors. Some particular solutions of (3.16) have already been pointed out in the literature [32]. We will extend these results in the next section.

(5) $J+a_1M^{(1)}+a_2M^{(2)}, P_z+b_1M^{(1)}+b_2M^{(2)}$

The reduction associated with the above subalgebra is

$$\psi^{(i)} = f^{(i)}(r) \exp[\sqrt{-1}\,\varepsilon^{i-1}(a_i\theta - b_i z)] \tag{3.17}$$

where

$$f^{(i)}_{rr} + \frac{1}{r} f^{(i)}_{r} + \left(b_i - \frac{a_1^2}{r^2}\right) f^{(i)} + \eta[|f^{(i)}|^2 + (1+h)|f^{(3-i)}|^2] f^{(i)} = 0. \tag{3.18}$$

For $f^{(i)}$ real, this leads to the stationary self-trapping solutions of (1.1). Some particular solutions for the uncoupled case have been extensively analysed in the literature [41-43]. These solutions are known to be unstable and eventually to collapse in the medium [38-40]. In analogy with the uncoupled case, the localized solutions of (3.18) can be obtained from the particular case where $b_i \to -\infty$ in (3.16) [27].

(6) $D+cJ+b_1M^{(1)}+b_2M^{(2)}, P_z$

This subalgebra leads to a logarithmic spiral-like symmetry variable

$$\xi = \theta + c \ln r \tag{3.19}$$

with the reduction

$$\psi^{(i)} = r^{-1} f^{(i)}(\xi) \exp\left[-\sqrt{-1}\,\varepsilon^{i-1} \frac{b_i}{2} \ln r\right] \tag{3.20}$$

where

$$(1+c^2) f^{(i)}_{\xi\xi} - 2c\left(1+\sqrt{-1}\,\frac{b_i}{2}\right) f^{(i)}_{\xi} + \left(1+\sqrt{-1}\,\frac{b_i}{2}\right)^2 f^{(i)} + \eta[|f^{(i)}|^2 + (1+h)|f^{(3-i)}|^2] f^{(i)} = 0. \tag{3.21}$$

Solutions of (3.21) are z-independent and singular at $r=0$. As such, they could be relevant in the filamentation theory [44]. However, one has to check for θ-periodicity [17, 24, 45].

Substituting (3.3) in (3.21), one obtains

$$\varphi_\xi^{(i)}=\frac{1}{[A^{(i)}]^2}\exp\left[\frac{2c\xi}{1+c^2}\right]\left\{S_i+\frac{cb_i}{2(1+c^2)}[A^{(i)}]^2\exp\left[\frac{-2c\xi}{1+c^2}\right]\right.$$
$$\left.+\frac{b_i}{(1+c^2)^2}\int[A^{(i)}]^2\exp\left[\frac{-2c\xi}{1+c^2}\right]\mathrm{d}\xi\right\} \tag{3.22}$$

where S_i are integration constants. The quantities $Y^{(i)}=\int[A^{(i)}]^2\,\mathrm{d}\xi$ satisfy two coupled third-order ODEs that reduce to second order when $b_i=0$, that is

$$(1+c^2)A_{\xi\xi}^{(i)}-(1+c^2)\frac{S_i^2}{[A^{(i)}]^3}\exp\left[\frac{4c\xi}{1+c^2}\right]-2cA_\xi^{(i)}+A^{(i)}$$
$$+\eta[(A^{(i)})^2+(1+h)(A^{(3-i)})^2]A^{(i)}=0. \tag{3.23}$$

For $S_i=0$, (3.23) describes two nonlinear coupled oscillators with damping for $c<0$. However, no single-valued solution in the x–y plane exists for that case [17, 24, 45].

For $c=b_i=0$ and $f^{(i)}$ real, exact analytical solutions of (3.21) that are periodic in the x–y plane can be found for identical fields $\psi^{(i)}$. These are

$$\psi^{(1)}=\psi^{(2)}=\frac{k}{r}\left[\frac{2}{\eta(2+h)(1-2k^2)}\right]^{1/2}\mathrm{cn}\left[\frac{\theta}{(1-2k^2)^{1/2}},k\right] \tag{3.24}$$

where $0<k^2<\frac{1}{2}$ is the squared-modulus of the cosine elliptic function $\mathrm{cn}(\,,k)$ and $h>-2(h<-2)$ for $\eta=1(\eta=-1)$. Periodic solutions correspond to values of k that satisfy

$$K(k)=\frac{\pi}{2n(1-2k^2)^{1/2}}\qquad n=2,3,4\ldots \tag{3.25}$$

where K is the complete elliptic integral of the first kind. For instance, the first three sets of values are ($k=0.635$; $n=2$), ($k=0.677$; $n=3$) and ($k=0.691$; $n=4$). The case $n=1$ gives $k=0$ and corresponds to the linear limit.

Figure 1 shows the field intensity contours for the first value of k. The fact that these solutions exist only for specific values of the modulus could make them relevant for the process of spatial pattern formation in nonlinear optics (see, for instance, [1] and references therein).

(7) K_x, K_y+cP_y

This subalgebra reduces (1.1) to coupled first-order ODEs for which the solution is

$$\psi^{(i)}=\frac{C_i}{\sqrt{z(z+c)}}\exp\left\{\sqrt{-1}\,\varepsilon^{i-1}\left[\frac{1}{4}\left(\frac{x^2}{z}+\frac{y^2}{z+c}\right)+\varphi^{(i)}(z)\right]\right\} \tag{3.26}$$

where C_i are constants and

$$\varphi^{(i)}=\eta[C_i^2+(1+h)C_{3-i}^2]\begin{cases}1/c\ln(z+c)/z & c\neq 0\\ 1/z & c=0.\end{cases} \tag{3.27}$$

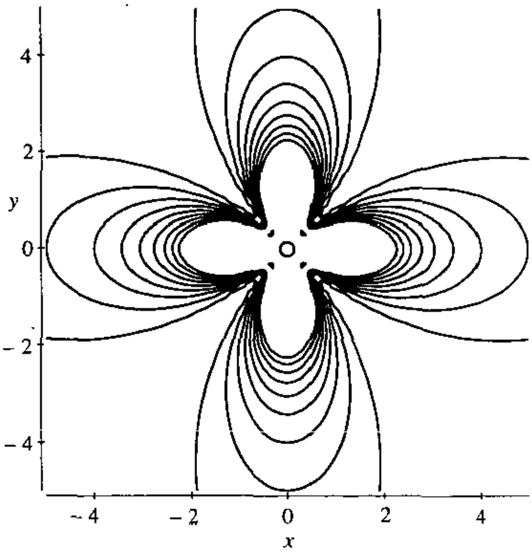

Figure 1. Contour plot of (3.24) for $k=0.635$ ($n=2$).

This solution physically describes an elliptical catastrophic self-focusing process because of its singularity behaviour at $z=0$ and $z=-c$ [38, 39]. A symmetric solution is obtained under the symmetry transformation $z \to z-c/2$.

(8) $D+b_1M^{(1)}+b_2M^{(2)}, K_y$

This subalgebra provides the symmetry variable

$$\xi = xz^{-1/2} \tag{3.28}$$

and the reduction

$$\psi^{(i)} = f^{(i)}(\xi)z^{-1/2}\exp\left[\sqrt{-1}\,\varepsilon^{i-1}\left(\frac{1}{4}\frac{y^2}{z} - b_i \ln z\right)\right] \tag{3.29}$$

where

$$f^{(i)}_{\xi\xi} + \frac{\sqrt{-1}}{2}\xi f^{(i)}_{\xi} + b_i f^{(i)} + \eta[|f^{(i)}|^2 + (1+h)|f^{(3-i)}|^2]f^{(i)} = 0. \tag{3.30}$$

Substituting (3.3) in (3.30) gives third-order coupled ODEs for $Y^{(i)} = \int [A^{(i)}]^2\,d\xi$, the solution of which can be given in terms of the Painlevé transcendent function for the uncoupled case [21]. We will not go further into the analysis of the generic coupled case.

4. Physical application of reduction (3.16)

The coefficients $\alpha^{(i)}(x, y, z) = (1+z^2)^{-1/2}\exp[\sqrt{-1}\,\varepsilon^{i-1}(z\xi^2/4 + a_i\theta - b_i\tan^{-1}z)]$ in (3.15) remind us of the diffraction law of Gaussian optical beams. In fact, in the linear limit $\eta=0$ the modal solutions of (3.16) are the Laguerre–Gauss functions. These solutions also describe the modes of a linear cavity with spherical resonators, that is a Fabry–Perot interferometer with spherical mirrors [46].

The aim of this section is to solve approximately the full nonlinear equation (3.16) in order to obtain the nonlinear generalization of these self-similar linear modes. The case of a cavity with plane mirrors, described by reduction (3.17), corresponds to the limiting case where $b_i \to -\infty$ in (3.16). These results generalize the analysis done in [32] for the fundamental mode of the cavity.

First, let us note that the reduction (3.15) can be embedded in the more general transformation [27]

$$\psi^{(i)} = (1+z^2)^{-1/2} U^{(i)}(Z, \xi) \exp[\sqrt{-1}\, \varepsilon^{i-1}(\tfrac{1}{4}z\xi^2 + a_i\theta)] \tag{4.1}$$

where $Z = \tan^{-1}(z)$ and $U^{(i)}(Z, \xi)$ are complex functions satisfying

$$\begin{aligned} &\sqrt{-1}\, U_Z^{(1)} + U_{\xi\xi}^{(1)} + \frac{1}{\xi} U_\xi^{(1)} - \tfrac{1}{4}\xi^2 U^{(1)} - \frac{a_1^2}{\xi^2} U^{(1)} + \eta[|U^{(1)}|^2 + (1+h)|U^{(2)}|^2]U^{(1)} = 0 \\ &\varepsilon\sqrt{-1}\, U_Z^{(2)} + U_{\xi\xi}^{(2)} + \frac{1}{\xi} U_\xi^{(2)} - \tfrac{1}{4}\xi^2 U^{(2)} - \frac{a_2^2}{\xi^2} U^{(2)} + \eta[|U^{(2)}|^2 + (1+h)|U^{(1)}|^2]U^{(2)} = 0. \end{aligned} \tag{4.2}$$

Equations (4.2) describe the nonlinear propagation of two cylindrical waves in a nonlinear Kerr medium with parabolic refractive index profile. This system has stationary solutions of the form

$$U^{(i)} = f^{(i)}(\xi) \exp[-\sqrt{-1}\, \varepsilon^{i-1} b_i Z] \tag{4.3}$$

with $f^{(i)}(\xi)$ satisfying equation (3.16). Thus, transformation (4.1) establishes a link between radiative-like solution (3.15) and stationary solution (4.3). Since the variational analysis is more easily tractable on the basis of stationary solutions, we will retain (4.2) as our basic evolution equation. As an additional result, we will obtain an approximate analytical description of the quasistationary propagation of (4.1), which is not without value.

Equation (4.2) can be reformulated as a variational problem with the Lagrangian

$$\Lambda = \Lambda^{(1)} + \Lambda^{(2)} + \Lambda^{(12)} \tag{4.4}$$

where

$$\Lambda^{(i)} = \varepsilon^{i-1} \frac{\sqrt{-1}}{2} [U^{(i)} U_Z^{(i)*} - U^{(i)*} U_Z^{(i)}] + |U_\xi^{(i)}|^2 + \tfrac{1}{4}\xi^2 |U^{(i)}|^2 + \frac{a_i^2}{\xi^2} |U^{(i)}|^2 - \tfrac{1}{2}\eta |U^{(i)}|^4 \tag{4.5}$$

and

$$\Lambda^{(12)} = -\eta(1+h)|U^{(1)}|^2 |U^{(2)}|^2. \tag{4.6}$$

Equation (4.2) is then derived from the cylindrical Euler–Lagrange equations

$$\frac{\partial}{\partial Z}\left[\frac{\partial \Lambda}{\partial U_Z^{(i)*}}\right] + \frac{\partial}{\partial \xi}\left[\frac{\partial \Lambda}{\partial U_\xi^{(i)*}}\right] + \frac{1}{\xi}\frac{\partial \Lambda}{\partial U_\xi^{(i)*}} - \frac{\partial \Lambda}{\partial U^{(i)*}} = 0. \tag{4.7}$$

The essence of the variational approach lies in the choice of the most appropriate trial functions that describe, as faithfully as possible, the exact self-similar solutions behaviour. On the other hand, since we want to obtain analytical results, we have to restrict our choice to a generic one. We found that a good compromise between simplicity and accuracy is given by the trial functions

$$U^{(i)} = A_i L^{(i)}\left(\frac{\xi}{W_i}\right) \exp[\sqrt{-1}\, \varepsilon^{i-1}(\varphi_i + B_i\xi^2)] \tag{4.8}$$

where parameters A_i, W_i, φ_i and B_i are all real functions of the variable Z. The choice of the quadratic phase variation $B_i\xi^2$ is a standard one in optical wave propagation and is necessary for those interested in a description of the quasistationary propagation [27, 32, 34-36]. Finally, we are reminded that the choice of real functions $L^{(i)}(\zeta_i)$, $\zeta_i = \xi/W_i$, is based on the form of the exact localized solutions of (3.16) in the linear limit $\eta=0$, which are the Laguerre-Gauss polynomials. In the following, we will restrict ourselves to the first four polynomials, that is

$$\begin{aligned} &L_1^{(i)} = \exp[-\zeta_i^2] && a_i = 0 \\ &L_2^{(i)} = \zeta_i \exp[-\zeta_i^2] && a_i = \pm 1 \\ &L_3^{(i)} = (1-2\zeta_i^2)\exp[-\zeta_i^2] && a_i = 0 \\ &L_4^{(i)} = \zeta_i^2 \exp[-\zeta_i^2] && a_i = \pm 2. \end{aligned} \tag{4.9}$$

Substituting the ansatz (4.8)-(4.9) in the Lagrangian (4.4) and integrating the ξ-variable from 0 to infinity yield the reduced (or averaged) Lagrangian

$$\langle\Lambda\rangle = \langle\Lambda^{(1)}\rangle + \langle\Lambda^{(2)}\rangle + \langle\Lambda^{(12)}\rangle \tag{4.10}$$

where

$$\begin{aligned} \langle\Lambda^{(i)}\rangle = A_i^2W_i^2\dot{\varphi}_i\alpha_{1i} + A_i^2\dot{B}_iW_i^4\alpha_{2i} + A_i^2\alpha_{3i} + 4A_i^2B_i^2W_i^4\alpha_{2i} \\ + A_i^2a_i^2\alpha_{4i} + \tfrac{1}{4}A_i^2W_i^4\alpha_{2i} - \tfrac{1}{2}\eta A_i^4W_i^2\alpha_{5i} \end{aligned} \tag{4.11}$$

and

$$\langle\Lambda^{(12)}\rangle = -\eta(1+h)A_1^2A_2^2\alpha_6. \tag{4.12}$$

Throughout the text, the dot means derivative with respect to Z. The coefficients α_{ki} $(k=1,\ldots,5)$ and α_6 are given by

$$\begin{aligned} &\alpha_{1i} = \int_0^\infty [L^{(i)}]^2\zeta_i\,\mathrm{d}\zeta_i && \alpha_{2i} = \int_0^\infty [L^{(i)}]^2\zeta_i^3\,\mathrm{d}\zeta_i \\ &\alpha_{3i} = \int_0^\infty \left[\frac{\mathrm{d}L^{(i)}}{\mathrm{d}\zeta_i}\right]^2\zeta_i\,\mathrm{d}\zeta_i && \alpha_{4i} = \int_0^\infty [L^{(i)}]^2\zeta_i^{-1}\,\mathrm{d}\zeta_i \\ &\alpha_{5i} = \int_0^\infty [L^{(i)}]^4\zeta_i\,\mathrm{d}\zeta_i && \alpha_6 = \int_0^\infty \left[L^{(1)}\left(\frac{\xi}{W_1}\right)\right]^2\left[L^{(2)}\left(\frac{\xi}{W_2}\right)\right]^2\xi\,\mathrm{d}\xi \end{aligned} \tag{4.13}$$

and can be evaluated analytically. Their values are summarized in table 2.

The reduced Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}Z}\left[\frac{\partial\langle\Lambda\rangle}{\partial y_{iZ}}\right] = \frac{\partial\langle\Lambda\rangle}{\partial y_i} \tag{4.14}$$

Table 2. Values of α_k $(k=1,2,3,4,5)$ for the first four Laguerre polynomials.

	L_1	L_2	L_3	L_4
α_1	1/4	1/8	1/4	1/8
α_2	1/8	1/8	3/8	3/16
α_3	1/2	1/4	3/2	1/4
α_4		1/4		1/8
α_5	1/8	1/64	1/16	3/256

where y_i stand for A_i, W_i, φ_i and B_i provide the eight following relations:

$$\frac{\delta\langle\Lambda\rangle}{\partial A_i}=0\Rightarrow W_i^2\dot{\varphi}_i\alpha_{1i}+\dot{B}_iW_i^4\alpha_{2i}+\alpha_{3i}+4B_i^2W_i^4\alpha_{2i}+a_i^2\alpha_{4i}$$

$$+\frac{1}{4}W_i^4\alpha_{2i}-\eta A_i^2W_i^2\alpha_{5i}-\eta(1+h)A_{3-i}^2\alpha_6=0 \tag{4.15}$$

$$\frac{\delta\langle\Lambda\rangle}{\delta\varphi_i}=0\Rightarrow A_i^2W_i^2=E_i=\text{constants} \tag{4.16}$$

$$\frac{\delta\langle\Lambda\rangle}{\delta B_i}=0\Rightarrow \dot{W}_i=4B_iW_i \tag{4.17}$$

$$\frac{\delta\langle\Lambda\rangle}{\delta W_i}=0\Rightarrow 2W_i^2\dot{\varphi}_i\alpha_{1i}+4\dot{B}_iW_i^4\alpha_{2i}+16B_i^2W_i^4\alpha_{2i}+W_i^4\alpha_{2i}$$

$$-\eta A_i^2W_i^2\alpha_{5i}-\eta(1+h)A_{3-i}^2W_i\frac{\mathrm{d}\alpha_6}{\mathrm{d}W_i}=0 \tag{4.18}$$

where the constants E_i are proportional to the energy Σ_i in each wave through

$$\Sigma_i=\int_0^{2\pi}\int_0^\infty|\psi^{(i)}|^2r\,\mathrm{d}r\,\mathrm{d}\theta=2\pi\int_0^\infty|U^{(i)}|^2\xi\,\mathrm{d}\xi=2\pi\alpha_{1i}E_i. \tag{4.19}$$

Note that there is no energy exchange between the waves during the propagation. Relations (4.15)-(4.18) give an approximate description of the evolution of the fields $U^{(i)}$ around the stationary solution (4.3) of (4.2). We will not attempt to solve them explicitly but rather restrict our analysis to the stationary case.

The determination and classification of localized non-singular solutions of (3.16) are then reduced to the determination of the relations between the amplitudes A_i (or E_i) and widths W_i in (4.8) when A_i and W_i are assumed constants, $\varphi_i=-b_iZ$ and $B_i=0$. Substituting these values in (4.15), (4.16) and (4.18), one obtains

$$E_i+\frac{(1+h)}{\alpha_{5i}W_{3-i}^2}\left[2\alpha_6-W_i\frac{\mathrm{d}\alpha_6}{\mathrm{d}W_i}\right]E_{3-i}=\frac{\eta}{\alpha_{5i}}[2\alpha_{3i}+2a_i^2\alpha_{4i}-\tfrac{1}{2}W_i^4\alpha_{2i}] \tag{4.20}$$

and

$$b_i=\frac{1}{\alpha_{1i}W_i^2}\left[a_{3i}+a_i^2\alpha_{4i}+\tfrac{1}{4}W_i^4\alpha_{2i}-\eta E_i\alpha_{5i}-\eta(1+h)\frac{E_{3-i}}{W_{3-i}}\alpha_6\right]. \tag{4.21}$$

Relations (4.20) and (4.21) are parametric equations that give E_i and b_i as a function of the widths W_1 and W_2 of the approximate localized solutions of (3.16). We solved them for various values of W_1 and W_2.

Figure 2 shows the normalized energy $S=(2+h)\Sigma_i$ in each wave as a function of b for two identical beams, i.e. $A_1=A_2$, $W_1=W_2$ and $b_1=b_2=b$. The curve numbers refer to the mode number in (4.9). The signs + and − refer to a self-focusing medium ($\eta=1$) and a self-defocusing medium ($\eta=-1$) respectively. The points $b=1,2,3\ldots$ and $S=0$ correspond to the linear limit $\eta\to 0$. The case described by curve 1^+ was studied in [32]. The energy values at $b\to-\infty$ are $4\pi, 16\pi, 24\pi, 32\pi$ and correspond approximately to the energy of the first self-trapping solutions of (1.1) (no z-dependence in the amplitudes) [41-43]. The fundamental self-trapping solution is known to be unstable and eventually to collapse in a self-focusing process [38-40]. We suspect a

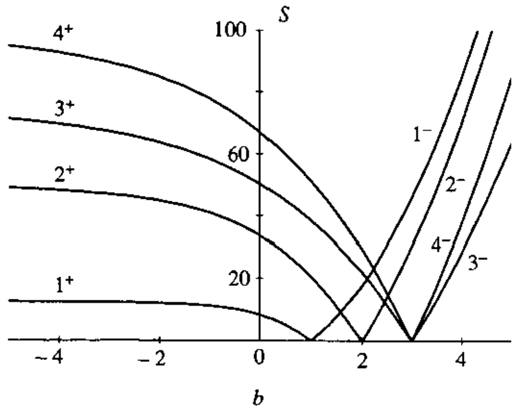

Figure 2. Normalized energy $S=(2+h)\ \Sigma_i$ in each wave for two identical beams. Curve numbers refer to $L_1, \ldots, L_4$ respectively while signs + and − stand for $\eta=1$ and $\eta=-1$.

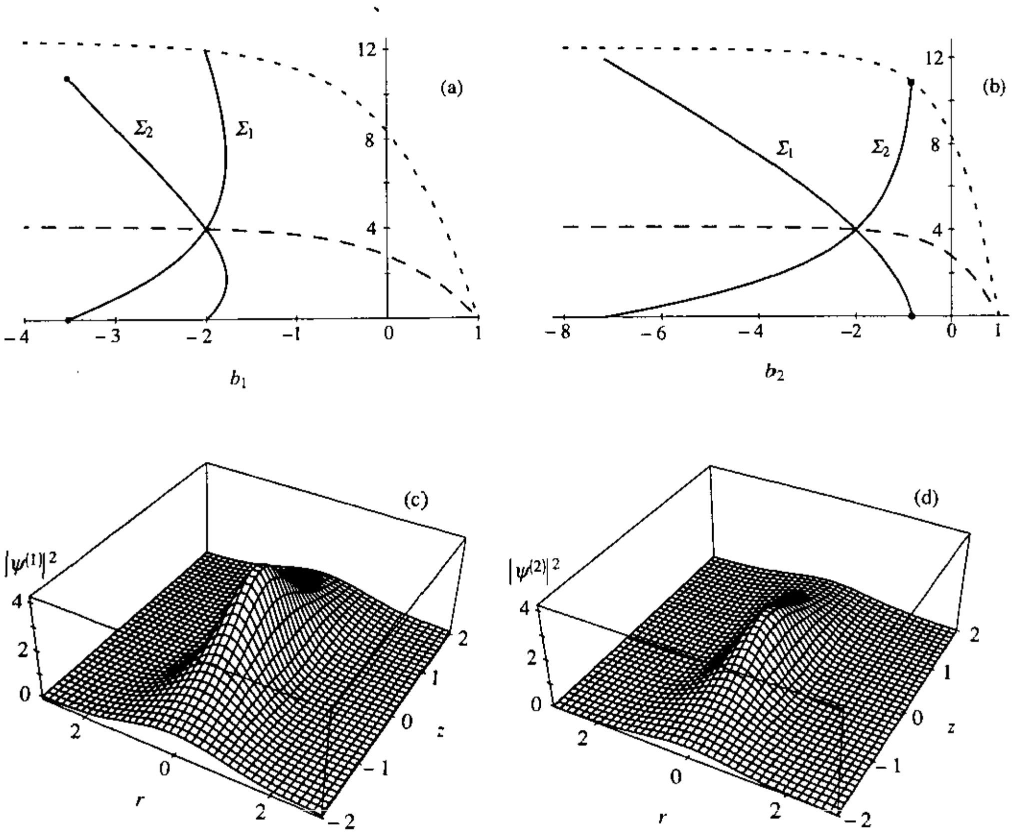

Figure 3. Energies Σ_1 and Σ_2 as a function of b_1 (a) and b_2 (b) for two beams in the fundamental mode with $\eta=1$, $h=1$, $W_1=0.928$ and $0.695 \leqslant W_2 \leqslant 1.209$. Intensity profile of the first (c) and second (d) beam for the parameter width $W_2=0.85$.

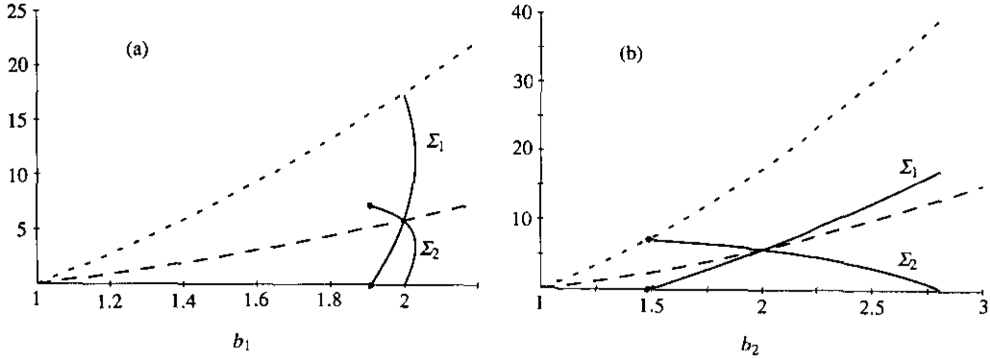

Figure 4. Energies Σ_1 and Σ_2 as a function of b_1 (a) and b_2 (b) for two beams in the fundamental mode with $\eta=-1$, $h=1$, $W_1=2.489$ and $2.240\leqslant W_2\leqslant 2.970$.

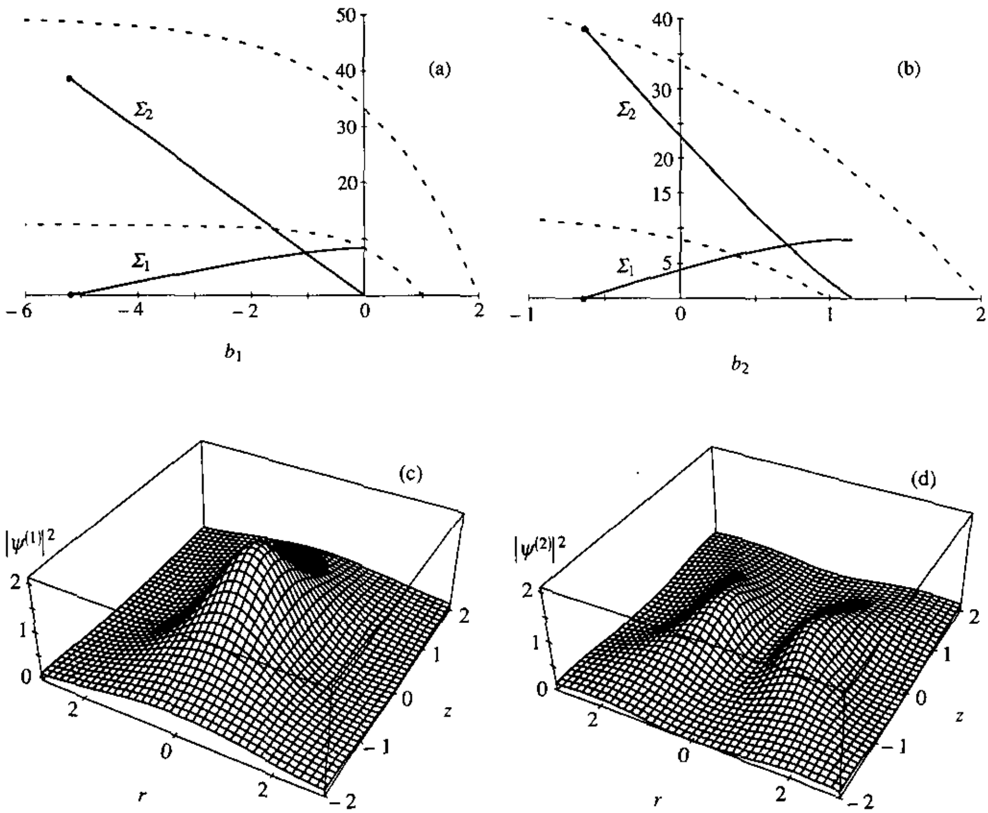

Figure 5. Energies Σ_1 and Σ_2 as a function of b_1 (a) and b_2 (b) for the first beam in the fundamental mode and the second beam in the second mode with $\eta=1$, $h=1$, $W_1=1.520$ and $1.387\leqslant W_2\leqslant 1.520$. Intensity profiles of the first (c) and second (d) beam for the parameter width $w_2=1.45$.

similar behaviour for the higher-order self-trapping solutions [47]. But here, we are more interested in the self-similar solutions which exist for energies below the critical self-trapping energies.

In addition to the above identical localized self-similar solutions, there is a relatively large set of solutions where $W_1 \neq W_2$. This physically corresponds to the case where the net flux within the cavity does not vanish [32]. For example, full curves on figures 3 and 4 show the energies Σ_1 and Σ_2 as a function of b_1 and b_2 for two beams in the fundamental mode with $h=1$. For comparison, the long broken curve give the energy for the case $W_1 = W_2$ while the short broken curve gives the energy for $h=-1$ (no nonlinear coupling). In figure 3 ($\eta=1$), we have chosen $W_1 = 0.928$ which leads to $0.695 \leqslant W_2 \leqslant 1.209$ and provides the possible self-similar solutions centred around $b_1 = b_2 = -2$. The points on the full curves correspond to $W_2 = 1.209$. As a typical example, we give the intensity profiles of the beams for $W_2 = 0.85$ on figures 3(c) and (d). In figure 4 ($\eta=-1$), we have set $W_1 = 2.489$ and $2.240 \leqslant W_2 \leqslant 2.970$ that correspond to a set of values centred around $b_1 = b_2 = 2$. Here, the points on the full curves are for $W_2 = 2.240$. An interesting point that comes out from figures 3 and 4 is that two different solutions exist for given parameters b_i which could provides the onset for bistability.

The most significant result of our analysis is the possible coexistence of self-similar beams having different modal profiles. For example, full curves on figure 5 show the energy values of the beams when the first is in the fundamental mode and the second in the second mode with $\eta=1$, $W_1 = 1.520$ and $1.387 \leqslant W_2 \leqslant 1.520$. The broken curves correspond to the energy in each beam without nonlinear coupling ($h=-1$). The points on the full curve are the limiting case $W_2 = 1.387$. A typical intensity profile is also shown on figures 5(c) and (d) for parameter width $W_2 = 1.45$.

The above results are only few examples from the possible self-similar nonlinear structures predicted by (1.1). It seems that the 'nonlinear superposition' of two beams having different modal profiles is always possible within a certain parameter range.

5. Discussion of the (1+1)-dimensional case

The (1+1)-dimensional version of (1.1) (no y-dependence for example) has the famous property of being completely integrable for $h=0$. It is identified as the Manakov equation and can be solved by the inverse scattering transform [48, 49]. This model is of particular interest in wave propagation in optical fibres (with x being a time-like variable). A general analysis of the travelling wave solutions for that case has been carried out in [50]. Unfortunately, this reduction of dimension breaks the conformal symmetry property present in the (2+1)-dimensional case. A consequence is the loss of the self-similar solutions (3.16). However, if the $|\psi^{(i)}|^2$ terms are replaced by (i) $(1+z^2)^{-1/2}|\psi^{(i)}|^2$ or by (ii) $|\psi^{(i)}|^4$ in (1.1) then a point-symmetry generated by the vector field

$$V = (1+z^2)P_z + zxP_x + zyP_y - \tfrac{1}{4}x^2(M^{(1)} + M^{(2)}) \\ - \tfrac{1}{2}z(\psi^{(1)}\partial_{\psi^{(1)}} + \psi^{(2)}\partial_{\psi^{(2)}} + \mathrm{cc}) + b_1 M^{(1)} + b_2 M^{(2)} \tag{5.1}$$

does lead to an equivalent reduction

$$\psi^{(i)} = f^{(i)}(\xi)(1+z^2)^{-1/4}\exp[\sqrt{-1}\,\varepsilon^{i-1}(\tfrac{1}{4}z\xi^2 - b_i\tan^{-1}z)] \\ \xi = x^2(1+z^2)^{-1} \tag{5.2}$$

where $f^{(i)}(\xi)$ satisfy

$$f^{(i)}_{\xi\xi}+(b_i-\tfrac{1}{4}\xi^2)f^{(i)}+\eta[|f^{(i)}|^2+(1+h)|f^{(3-i)}|^2]f^{(i)}=0. \tag{5.3}$$

The first case is still physically interesting from a phenomenological point of view as it simply redistributes the nonlinearity along the z-coordinate. Nonlinear modal properties that are equivalent to those described in section 4 do exist for the first case and are qualitatively the same as for the (2+1)-dimensional case except that the modal structure of the self-similar solutions are approximately given by the Hermite–Gauss polynomials [27]. In fact, the self-similar solutions (5.2) can be embedded into

$$\begin{aligned}\psi^{(i)}&=(1+z^2)^{-1/4}U^{(i)}(Z,\xi)\exp[\tfrac{1}{4}\sqrt{-1}\,\varepsilon^{i-1}z\xi^2]\\ \xi^2&=x^2(1+z^2)^{-1}\end{aligned} \tag{5.4}$$

where $U^{(i)}$ are the stationary solutions of

$$\begin{aligned}&\sqrt{-1}\,U^{(1)}_Z+U^{(1)}_{\xi\xi}-\tfrac{1}{4}\xi^2U^{(1)}+\eta[|U^{(1)}|^2+(1+h)|U^{(2)}|^2]U^{(1)}=0\\ &\varepsilon\sqrt{-1}\,U^{(2)}_Z+U^{(2)}_{\xi\xi}-\tfrac{1}{4}\xi^2U^{(2)}+\eta[|U^{(2)}|^2+(1+h)|U^{(1)}|^2]U^{(2)}=0.\end{aligned} \tag{5.5}$$

Equation (5.5) derives from the same Lagrangian (4.4)–(4.6) with $a_i=0$ and under the Euler–Lagrange equations

$$\frac{\partial}{\partial Z}\left[\frac{\partial\Lambda}{\partial U^{(i)*}_Z}\right]+\frac{\partial}{\partial\xi}\left[\frac{\partial\Lambda}{\partial U^{(i)*}_\xi}\right]-\frac{\partial\Lambda}{\partial U^{(i)*}}=0. \tag{5.6}$$

We can choose the same form as in (4.8) for the trial functions except that the Laguerre–Gauss polynomials L are replaced by the Hermite–Gauss polynomial H. The reduced Lagrangian and its corresponding Euler–Lagrange equations then lead to relations similar to (4.14)–(4.18), that is

$$\begin{aligned}W_i^2\dot\varphi_i c_{1i}+\dot B_iW_i^4c_{2i}+c_{3i}+4B_i^2W_i^4c_{2i}+\tfrac{1}{4}W_i^4c_{2i}-\eta A_i^2W_i^2c_{4i}&\\ -\eta(1+h)A_{3-i}^2W_ic_5=0&\end{aligned} \tag{5.7}$$

$$A_i^2W_i=E_i=\text{constants} \tag{5.8}$$

$$\dot W_i=4B_iW_i \tag{5.9}$$

$$\begin{aligned}W_i^2\dot\varphi_ic_{1i}+3\dot B_iW_i^4c_{2i}-c_{3i}+12B_i^2W_i^4c_{2i}+\tfrac{3}{4}W_i^4c_{2i}-\tfrac{1}{2}\eta A_i^2W_i^2c_{4i}&\\ -\eta(1+h)A_{3-i}^2\frac{\mathrm{d}c_5}{\mathrm{d}W_i}=0&\end{aligned} \tag{5.10}$$

where the constants E_i are related to Σ_i through

$$\Sigma_i=\int_{-\infty}^{\infty}|\psi^{(i)}|^2\,\mathrm{d}x=\int_{-\infty}^{\infty}|U^{(i)}|^2\,\mathrm{d}\xi=c_{1i}E_i. \tag{5.11}$$

The coefficients c_{ki} $(k=1,\ldots,4)$ and c_5 are given by

$$\begin{aligned}c_{1i}&=\int_{-\infty}^{\infty}[H^{(i)}]^2\,\mathrm{d}\zeta_i \qquad c_{2i}=\int_{-\infty}^{\infty}[H^{(i)}]^2\zeta_i^2\,\mathrm{d}\zeta_i\\ c_{3i}&=\int_{-\infty}^{\infty}\left[\frac{\mathrm{d}H^{(i)}}{\mathrm{d}\zeta_i}\right]^2\mathrm{d}\zeta_i \qquad c_{4i}=\int_{-\infty}^{\infty}[H^{(i)}]^4\,\mathrm{d}\zeta_i\\ c_5&=\int_{-\infty}^{\infty}\left[H^{(1)}\left(\frac{\xi}{W_1}\right)\right]^2\left[H^{(2)}\left(\frac{\xi}{W_2}\right)\right]^2\mathrm{d}\xi\end{aligned} \tag{5.12}$$

and evaluated analytically for the first two polynomials in table 3 (H_1 and H_2 are identical in form to L_1 and L_2).

Finally, the relations that determine the approximate parameter range of the localized self-similar solutions are

$$E_i + \frac{2(1+h)}{c_{4i} W_{3-i}} \left[c_5 - W_i \frac{\mathrm{d} c_5}{\mathrm{d} W_i} \right] E_{3-i} = \frac{\eta}{c_{4i} W_i} [4c_{3i} - W_i^4 c_{2i}] \tag{5.13}$$

and

$$b_i = \frac{1}{c_{1i} W_i^2} \left[c_{3i} + \tfrac{1}{4} W_i^4 c_{2i} - \eta E_i W_i c_{4i} - \eta (1+h) E_{3-i} \frac{W_i}{W_{3-i}} c_5 \right]. \tag{5.14}$$

As (4.20) and (4.21) relations (5.13) and (5.14) are parametric equations that give E_i and b_i as a function of the parameters widths W_1 and W_2. For instance, figure 6 shows the normalized energy $S = (2+h)\,\Sigma_i$ in each wave as a function of b for two identical beams. The notation is the same as in figure 2. The only major difference from figure 1 lies in the unsaturating behaviour of S at $b \to -\infty$. All other predictions and behaviours discussed for the (2+1)-dimensional case are also applicable here. Furthermore, because it is less demanding in computer time, this (1+1)-dimensional model can be useful to those interested in a numerical study of the self-similar solutions behaviour.

When the coupling terms in the (1+1)-dimensional model are a quartic nonlinearity as in the second example, the field equations do not derive from a Lagrangian; thus, the approximate variational method above cannot be applied.

Table 3. Values of c_k $(k = 1, 2, 3, 4)$ for the first two Hermite polynomials.

	H_1	H_2
c_1	$\sqrt{\pi/2}$	$(1/4)\sqrt{\pi/2}$
c_2	$(1/4)\sqrt{\pi/2}$	$(3/16)\sqrt{\pi/2}$
c_3	$\sqrt{\pi/2}$	$(3/4)\sqrt{\pi/2}$
c_4	$\sqrt{\pi/4}$	$(3/64)\sqrt{\pi/4}$

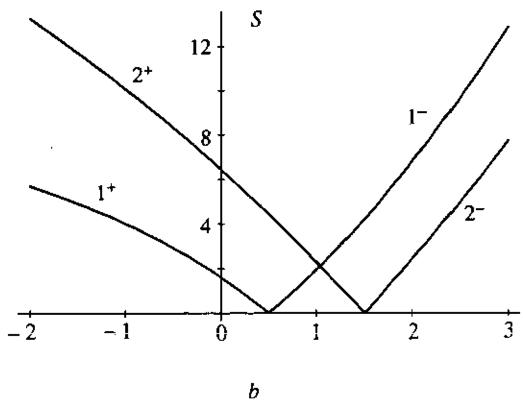

Figure 6. Normalized energy $S = (2+h)\,\Sigma_i$ in each wave for two identical beams. Curve numbers refer to H_1 and H_2 respectively while signs + and − refer to $\eta = 1$ and $\eta = -1$.

6. Conclusions

We have given the point-symmetry properties of the set of coupled nonlinear Schrödinger equations (1.1). The symmetry algebra g was shown to be a direct sum between the 2D Schrödinger algebra sch(2) and a constant change of phase generator. We have used a previous classification in conjugacy classes of the subalgebras of sch(2) to help in the determination of subalgebras of g. We have applied the symmetry reduction method for some of these subalgebras that lead to generic (2+1)-dimensional reductions. These solutions are of particular interest in the field of transverse effects in nonlinear optics.

As a physical application, we have obtained approximate analytical expressions for the localized self-similar solutions of the equation describing the modal properties of a nonlinear Fabry–Perot interferometer with spherical mirrors. Our analysis was based on a variational method that has been applied in various recent studies in optics. This permitted us to classify all localized solutions of the reduced equations in terms of their energies and widths. These solutions appear to have a well defined nonlinear modal structure given approximately by the Laguerre–Gauss polynomials. They exist for self-focusing as well as self-defocusing media.

The results of this study are indicative of a large set of self-similar nonlinear coherent structures predicted by the models (1.1). The coexistence of these self-similar coupled waves can be of interest in the study of various nonlinear systems and in particular for the transverse instability of counterpropagating optical beams due to transverse effects [4]. Many important questions remain to be addressed. For instance, the stability and bistability properties of the nonlinear modal structures described in section 4 have to be analysed (see, for instance, the recent results in [51]). This of course necessitates more exact results about the reduced ODEs. We plan to go back to that issue in the near future.

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